# Mathematical Analysis

third level

second course

Mathematics department

College of Education for Pure Sciences

**Dr. Nadia Nadhim**

**2021 - 2020**

Lecher 1:

# Continuity

Def: Let $(X, d_1)$ and $(Y, d_2)$ are tow metric space. And $S \subseteq X$ , $f: S \to Y$ , we say that $f(x)$ lends to S the **Limit L** as x lend S to a

**Example**: let $f: R \to R$, s.t $f(x) = 3x - 1$, prove that $\lim_{x\to 2} f(x) = 5$
Proof:

$L = 5$ , $a = 2$ $d_1(x, y) = d_2(x, y) = |x - y|$.
$\forall \epsilon > 0$, to find $\delta > 0$, s.t $d_2(f(x), 5) < \epsilon$, wherever $d_1(x, 2) < \delta$
$d_1(x, 2) = |x - 2| < \delta$ , $d_2(f(x), 5) = |f(x) - 5| = |3x - 1 - 5|$
$= |3x - 6| = 3|x - 2| < 3\,\delta$.
Choose $\delta = \frac{\epsilon}{3} \to |f(x) - 5\,| < 3\,\delta = 3\,.\frac{\epsilon}{3} = \epsilon$.
$\therefore d_2(f(x), 5) < \epsilon , \to \lim_{x\to 2} f(x) = 5$

كيف نبرهن عن وجود نهاية:

- عندما يعطى $\lim_{x\to a} f(x) = L$
- الخطوة الأساسية التي يجب الوصول اليها هي $d_2(f(x), L) < \epsilon$
- نأتي بالقيمتين $a$ , $L$ من السؤال المعطى
- نكون تطبيقين مسافة نسمي الأول $d_1(x, y)$ ونسمي الثاني $d_2(x, y)$ ومتساوين للبعض همها وكلاهما يساوي $|x - y|$
- نفرض ان $0 < \epsilon \; \forall$ لكي نجد $0 < \delta$ بحيث ان $d_2(f(x), L) < \epsilon$ وان $d_1(x, a) < \delta$
- تكون القيم داخل المطلق متساوية للتطبيقين بعد سحب عامل مشترك ع الاغلب من $d_2(f(x), L)$
- هذا المشترك نتخلص منه بفرض ان $\delta$ تساوي أي قيمة تمكننا التخلص من ذلك الثابت مع وجود $\epsilon$
- اذا كان $d_2(f(x), L) < \epsilon$ فان $\lim_{x\to a} f(x) = L$

Theorem: if f has a limit point L, then this limit is unique.

Properties of limits of functions:

Let $S \subseteq R^m$ , $f: S \to R^m$ , and $g: S \to r^m$ ; s.t $\lim_{x\to a} f(x) = A$ , $\lim_{x\to a} g(x) = B$ , then

1. $\lim_{x\to a}(f + g)(x) = A + B$
2. $\lim_{x\to a}(f\,.\,g) = A\,B$
3. $\lim_{x\to a}\left(\frac{f}{g}\right) = \frac{A}{B}$ $\;but\;\; B \neq 0, g(x) \neq 0$

Def: Let $(X, d_1)$ and $(Y, d_2)$ are tow metric space suppose that $S \subseteq X$ , $p \in S$ $and$ $f: s \to Y$ we say **f** continuous at $x = p$ **iff** $\forall \epsilon > 0$ , $\exists\, \delta > 0$ s.t $d_2\big(f(x), f(p)\big) < \epsilon$ whereever $d_1(x, p) < \delta$. If f is continuous at $x, \forall \in S$ , then f is continuous on S.

**Example**: prove that $f: R \to R$ , $f(x) = x^2$ is continuous , $\forall\, x \in X$ ,
$X = Y = R$ , $d_1 = d_2 = |x - y|$ ?

Solution:

Let $p \in R$ to prove f continuous at $x = p$ ?

Let $\epsilon > 0$, to find $\delta > 0$ s.t $|f(x) - f(p)| < \epsilon$, if $|x - p| < \delta$.

$|f(x) - f(p)| = |x^2 - p^2| = |(x - p)(x - p)| = |x - p||x + p|$ .

$\leq |x - p| \quad (|x| + |p|)$ …………………(1)

Since: $||x| - |p|| \leq |x - p| < \delta, \rightarrow ||x| - |p|| < \delta$.

$\rightarrow \; -\delta < |x| - |p| < \delta \rightarrow |x| - |p| < \delta \rightarrow |x| < \delta + |p|$ ……………..(2)

Put (2) in (1)

$|f(x) - f(p)| \leq |x - p|(|x| + |p|) < |p| + \delta + |p|)$

$< \delta(2|p| + \delta) \leq \delta(2|p| + 1)$

(since $0 < \delta \leq 1$) .choose $\delta = \min\left\{1, \frac{\epsilon}{2|p|+1}\right\}$

$\rightarrow |f(x) - f(p)| < \frac{\epsilon}{2|p|+1}.(2|p| + 1) = \epsilon$

$\therefore f$ is continuous at $x = p$ , $\forall\, p \in R$ , $\rightarrow f$ is continuous on R

كيف نبرهن عن استمرارية الدالة:

- الهدف هو الوصول الى $|f(x) - f(p)| < \epsilon$
- نفرض ان

$$Let\ p \in R\ T.P\ f\ continuous\ at\ x = p$$

- ونفرض أيضا

$$\epsilon > 0\ to\ find\ \delta > 0\ s.t\ |f(x) - f(p)| < \epsilon\ if\ |x - p| < \delta$$

- نحل المعادلة $|f(x) - f(p)|$ وباستخدام خواص المطلق نحصل معادلة رقم 1
- ومن خواص المطلق القيمة بين الموجب والسالب وبعزل $x$ في طرف وبقية المتغيرات في طرف اخر نحصل على معادلة 2
- نأخذ $min$ بين العدد الحقيقي $\epsilon$ على (مقام) القيمة المشتركة بين الطرفين من تعويض 2 في 1

Theorem: Let $(X, d_1)$ and $(Y, d_2)$ are two metric spaces $f: X \rightarrow Y$ be a function. Then $f$ is continuous at $p \in X$ iff $\lim_{n\rightarrow\infty} f(x_n) = f(p)$ for each sequence $< X_n >$ in $X$ and $X_n \rightarrow p$.

**Example**:

1- Let $f: R \to R$ s.t $f(x) = \begin{cases} 1 & if \ x > 0 \\ 0 & if \ x = 0 \\ -1 & if \ x < 0 \end{cases}$ use theorem to prove $f$ is not continuous at $p = 0$

Sol:

Let $\langle X_n \rangle = \langle \frac{1}{n} \rangle$ then $\frac{1}{n} \to 0 \ \ as \ \ n \to \ \infty \, \forall n \, , f(x) = f\left(\frac{1}{n}\right) = 1$ ,then

$f(x_n) = f\left(\frac{1}{n}\right) = 1 \, , 1 \, , \dots$ converge to $1 \neq 0 = f(0)$

by theorem above ( … ), $f$ is not continuous at $p = 0$.

2- Let $f: [a, b] \to R$ , s.t $f(x) = \begin{cases} 1 & if \ x \in Q \\ 2 & if \ x \in Q' \end{cases}$ use theorem above to prove that $f$ is not continuous

Sol:

Let $p \in [a, b] \to p \in Q \ or \ p \in Q'$

i. If $p \in Q$ since (between any two reals there are infinitely rationales and irrationals)
And (for every real $p$, there is irrational (or rational) Caushy seq. Converge to $p$)
$\Rightarrow$ there is an irrational Sequence $\langle X_n \rangle \ x_n \to p$.
$\Rightarrow f(x_n) = 2$ ,there $\langle f(x_n) \rangle = 2 \, ,2 \, , 2 \, , \dots$., Converge to $2 \neq 1 = f(p)$
by theorem above: ( … ), $f$ is not continuous at $p \ \forall p \in Q$.

ii. If $p \in Q'$,
by similar way we can show that $f$ is not continuous at $p, \forall \ p \ \ s.t \ p \in Q'$,
$\therefore f$ is not Continuous at $p, \forall p \in [a, b]$

Lecher 2:

Theorem: Let $(X, d_1)$ and $(Y, d_2)$ are two metric spaces, $f: X \to Y$ be a function, then $f$ is continuous on $X$ iff $f^{-1}(V)$ is open set in $X$, $\forall \ v$ is opens set in $Y$.

**Example**: Let $f: R \to R$ s.t $f(x) = x^2$. Use theorem above to prove that $f$ is Continuous on $R$?

Proof:

Let $V$ open set in $R$, then $V = (a, b)$ there are three case

1. If $a, b > 0 \to f^{-1}(V) = (\sqrt{a}, \sqrt{b}) \cup (-\sqrt{b}, -\sqrt{a})$ open
   $\therefore f^{-1}(V)$ is open in $X$
2. If $a < 0 \, , b > 0 \to f^{-1}(V) = (-\sqrt{b}, \sqrt{b})$ open
   $\therefore f^{-1}(V)$ is open in $X$
3. If $a, b < 0 \to f^{-1}(V) = \emptyset$ open $\to f^{-1}(V)$ is open in $X$
   $\therefore \forall \ V$ open in $Y \to f^{-1}(V)$ is open in $X \to f$ is continuous on $X$ by theorem: ( ... )

**Example**: Let $f: R \to R$ s.t $f(x) = \begin{cases} 1 & if\ x > 0 \\ 0 & if\ x = 0 \\ -1 & if\ x < 0 \end{cases}$. Use theorem above to prove that $f$ is not continuous at 0?

Proof:

Let $V = \left(-\frac{1}{2}, \frac{1}{2}\right)$ is open set in $Y$

Then $f^{-1}(V) = \{0\}$ is closed set in X

by theorem above $f$ is not continuous at $p = 0$

Theorem: Let $(X, d)$ be a matric space and $f, g$ are tow real valued function, if $f, g$ continuous, then

1. $f \pm g$
2. $f . g$
3. $rf$, $r \in R$
4. $\frac{f}{g}$, $g \neq 0$ are continuous

**Corollary**: every polynomial $p(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$ is continuous where $a_i$ is constant.

Proof:

Since $f(x) = x^n$ is continuous function, $\forall n$ and $rf$ is Continuous be theorem, then the sum of Continuous function is continuous by theorem above.

# The Properties of Real Valued Function Defined on Compact Spaces

Theorem: Let $(X, d_1)$ and $(Y, d_2)$ are two metric space and $f: X \to Y$ continuous function, if $X$ is compact, then $f(x)$ is compact.

Proof:

$f(x) = \{ f(x) : x \in X \}$, Let $\{V_\lambda : \lambda \in \Lambda \}$ be an open set in $Y$, $\forall \lambda \in \Lambda$ and $f$ is Continuous then $f^{-1}(V_\lambda)$ is open set in $X$,$\forall \lambda \in \Lambda$ (by theorem)

Since $f(x) \subseteq \bigcup_{\forall \lambda \in \Lambda} V_\lambda$

$\to X \subseteq f^{-1}(\bigcup_{\lambda \in \Lambda} V_\lambda) = \bigcup_{\lambda \in \Lambda} \left(f^{-1}(V_\lambda)\right)$

$\to \{f^{-1}(V_\lambda) : \lambda \in \Lambda \}$ is open cover for X.

$\to$ there is finite subcover $\{f^{-1}(V_1), f^{-1}(V_2), \ldots, f^{-1}(V_n) \}$ for X (since X is compact)

$\to X \subseteq \bigcup_{i=1}^{n} f^{-1}(V_i) = f^{-1}(\bigcup_{i=1}^{n} V_i)$

$\to X \subseteq f^{-1}(\bigcup_{i=1}^{n} V_i)$

$\to f(x) \subseteq \bigcup_{i=1}^{n} V_i$

$\to \{V_1, V_2, \ldots, V_n\}$ is finite subcover for $f(x)$

$\therefore f(x)$ is compact

Lecher 3:

Definition: Let $(X,d)$ be a metric space and $f: X \to R$ we say that $f$ is bounded if $\exists\ M > 0$, s.t $|f(x)| \le M, \forall x \in X$.

i.e.: $f(x) = \{f(x) : x \in X\}$ is bounded set if has upper and lower bonded

i.e.: $f$ is bounded iff $f(x)$ is bounded set.

Theorem: Let $(X,d)$ be a metric space and $f: X \to R$ continuous function, if X is compact, then $f$ is bounded.

Proof:

Since $f$ is continuous and $X$ is compact

Then $f(x)$ is compact by theorem.

$\to f(x)$ is closed and bounded.

$\to f$ is bounded.

**Example**: Give example for bonded function and its domain not compact.

Sol:

$X = (0,1)$ not compact, $f: X \to R, f(x) = 3x$ if $M = 3 \to |f(x)| \le 3$, then $f$ is bounded.

Remark: the condition of compactness in above theorem is necessary.

**Example**: Let $X = (0,\infty)$ not compact, $f(x) = \frac{1}{x}$ continuous, $X > c\ f$ is not bounded since $\forall M > 0, \exists k \in N\ s.t\ M < \frac{1}{k} = f(x)$

$\forall x \in X, \exists M > 0\ \ s.t\ M|f(x)| \le M$

(by Arch. Prop.)

Definition: Let $(X,d)$ be a metric space and $f: X \to R$ bounded function, a point a is called.

1. Maximum extreme point of f if $f(x) \le f(a), \forall x \in X$
2. Minimum extreme point of f if $f(a) \le f(x), \forall x \in X$

Theorem: Let $f: X \to R$ be a continuous function, if X is compact, then $\exists a, b \in X$, such that $f(a) \le f(x) \le f(b), \forall x \in X$

i.e.: $f$ has min. extreme point at a, and f has max. extreme point at b.

When you see ***Compact*** remember

1. Doman function is **closed** interval
2. If **X** is compact then $f(x)$ is compact
3. If X is compact then $f$ is **bounded**
4. If $f(x)$ is compact then $f(x)$ is **closed** and **bounded**

Lecher 4:

# Uniformly continuity

Definition: Let $(X, d)$ be a metric space and $f: X \to R$, f is called uniformly continuity if

$$\forall \epsilon > 0, \exists \delta > 0 \; s.t \; |f(x) - f(y)| < \epsilon \; wherever \; d(x, y) < \delta, \forall x, y \in X$$

Remark: The choose of δ in the definition of uniformly continuous is depending on ϵ only

Theorem: every uniformly continuous is continuous.

Proof:

Let f be a uniformly continuous on X, then $\forall \epsilon > 0, \exists \delta > 0 \; s.t \; |f(x) - f(y)| < \epsilon$ wherever $d(x, y) < \delta, \forall x, y \in X$

$\to take \; y = p \to \; \forall \epsilon > 0, \exists \delta > 0 \; s.t \; |f(x) - f(p)| < \epsilon$ wherever $d(x, p) < \delta, \forall x, p \in X$, then f is continuous at $p, \forall p$.

Remark: The converse of theorem above is not true for example:

**Example:** Let $f: R \to R, \; f(x) = x^2$ is continuous function let $x = n, y = n + \frac{1}{n}, n \in N$

$d(x, y) = |x - y| = \left|n - n - \frac{1}{n}\right| = \left|-\frac{1}{n}\right| = \frac{1}{n} < \delta.$

By Arch. Prop. Any real δ ,$\exists n \in N \; s.t \frac{1}{n} < \delta$.

Take $\epsilon = 1 \to |f(x) - f(y)| = \left|n^2 - \left(n + \frac{1}{n}\right)^2\right| = \left|n^2 - n^2 - 2 - \frac{1}{n^2}\right|$

$$= 2 + \frac{1}{n^2} > \epsilon = 1$$

∴ f is not uniformly continuous.

Theorem: Let $f: X \to R$ be a continuous function. If X is compact, then f is uniformly continuous.

**Example:** Let $f: [a, b] \to R, f(x) = 3x$ is continuous then f is uniformly continuous.

Any closed interval $A = [a, b]$ is Compact

Theorem: (Intermediate Value property) (**I.V.P**)

Let f be a continuous on $[a, b]$ and $f(a) = \alpha, f(b) = \beta$, then for all $\gamma, \alpha < \gamma < \beta$, $\exists c, a < c < b$ and $f(c) = \gamma$

Proof:

Let $S = \{x: x \in [a, b], f(x) \le \gamma\}$.

$S \neq \emptyset$, Since $a \in S$, S is bounded above (Since b is upper bounded of S)

$\to$ by Completeness axiom, S has sup. Let $\sup(S) = c$.

Then there are three cases $f(c) = \gamma \; or \; f(c) < \gamma \; or \; f(c) > \gamma$

If $f(c) < \gamma$:

Since f is continuous at c and $f(c) < \gamma \to \; \exists \epsilon > 0$ s.t $f(x) < \gamma, \forall x \in (c - \epsilon, c + \epsilon)$

If $c < x < b \to x \in S$ and $c < x \to !$

$\to f(c) \not< \gamma$ (since $\sup(S) = c$)

If $f(c) > \gamma$, by Similar way $f(c) \not> \gamma$

$\to f(c) = \gamma \to \exists c, a < c < b$ and $f(c) = \gamma$

Theorem: (**Interval theorem**)

If f is continuous on $I = [a, b]$, then $f(I)$ is closed bounded interval.

Proof:

Since I is closed and bounded, then I is compact.

Since f is continuous on I then f has max , min extreme point $\to \exists c, d, f(c) = m$, $f(d) = M \quad s.t \; m \le f(x) \le M, \forall x$

There are two cases: $c < d \;\; or \;\; d < c$

If $c < d$, Apply (I.V.P) on f and $[c, d]$

$\to \forall y, y \in (m, M), \exists x \in (c, d) \; s.t \; f(x) = y$

$\therefore f(I) = [m, M]$

Theorem: (**Fixed point theorem**)

Let $f: [0,1] \to [0,1]$ continuous, then there is at least one number c (c is called fixed point) such that $f(c) = c$

Proof:

Suppose $g[0,1] \to R$, s.t $g(x) = f(x) - x$ continuous on [0,1] (since the identity function $c: X \to X$ is continuous and f continuous, then $g$ is continuous on [0,1] ).

If $f(0) = 0 \;\; or \;\; f(1) = 1$, then proof is complete.

Suppose $f(0) \neq 0 \;\; and \;\; f(1) \neq 1$

Since f is onto $(f: [0,1] \to [0,1]) \to g(0) = f(0) - 0 = f(0) > 0$

and $g(1) = f(1) - 1 < 0 \to g(1) < 0 < g(0)$

by (I.V.P), $\exists c, \; 0 < c < 1 \; s.t \; g(c) = 0 \to f(c) - c = 0 \to f(c) = c$

$\therefore$ f has a fixed point

Theorem: suppose $f: I \to J$ is bijective, where I, J are closed interval, if f is continuous, then $f^{-1}$ continuous

Theorem: Any polynomial of odd degree has at least one real root.

Lecher 5:

## Chapter Six

# The Differentiation

Definition: A function $f: A \to Y$ , $A \subseteq R$, is called differentiable at c iff $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists.
The value of this limit is called the derivative of $f$ at c and denoted by $f'(c)$

Theorem: if $f$ id differentiable at point c, then it is continuous at c

Proof:

Let $\epsilon > 0$, T.P $|f(x) - f(c)| < \epsilon$ ?

Since f is diff., then $f'(c) = \lim \frac{f(x)-f(c)}{x-c}$ exists

$$\lim_{x\to c}\big(f(x) - f(c)\big) = \lim_{x\to c}\left(f(x) - f(c)\ \frac{x-c}{x-c}\right)$$
$$= \lim \frac{f(x)-f(c)}{x-c} . \lim_{x\to c}(x-c)$$
$$= f'(c) = 0$$

So, $\lim_{x\to c}\big(f(x) - f(c)\big) = 0$, then $|f(x) - f(c)| < \epsilon$

Hence f is continuous at c.

**Where came this $f(x) - f(c)\ \frac{x-c}{x-c}$**

We know that def. of Continuous is $|f(x) - f(c)|$ and in same time we know that def. of diff. is $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ by put the tow def. $|f(x) - f(c)| = \frac{f(x)-f(c)}{x-c}$ but not equal we can be it is equal by multiplication $\frac{x-c}{x-c}$ on the right side. It is will be as $f(x) - f(c) = f(x) - f(c)\ \frac{x-c}{x-c}$

After that take limit, both said

Remark: The converse of the above theorem is not true

**Example:** Let $f: R \to R$, $f(x) = |x|$ continuous, but not diff at $c = 0$

Since, $f'(0) = \lim_{x\to \epsilon} \frac{f(x)-f(0)}{x-\epsilon} = \lim_{x\to 0} \frac{|x|-0}{x-0} = \lim_{x\to 0} \frac{|x|}{x} = \pm\frac{x}{x} = \pm 1$, not exists

Hence $f'(0)$ is not exists

So, f is not diff at $c = 0$

Limit value is **unique** except that does not exist

Theorem: (Rules of derivative)

Let f , g be two diff function at c, then

1. $(f + g)$ is diff at c and $(f + g)'(c) = f'(c) + g'(c)$
2. $(f \,.\, g)$ is diff at c and $(f \,.\, g)'(c) = f(c)g'(c) + g(c)f'(c)$
3. If $f(c) \neq 0$, then $\frac{1}{f(c)}$ is diff at c and $\left(\frac{1}{f}\right)'(c) = \frac{-f'(c)}{(f(c))^2}$
4. If $g(c) \neq 0$, then $\frac{f}{g}$ is diff at c and $\left(\frac{f}{g}\right)'(c) = \frac{g.f' - f.g'}{(g(c))^2}$
5. If , $\forall n \in N, f(x) = x^n$, then $f'(x) = n\, x^{n-1}$

Theorem: (Chain Rule)

Let f be diff function at c and g diff at $b = f(c)$ then $gof$ is diff at c and

$$(gof)'(c) = g'\big(f(c)\big).\, f'(c)$$

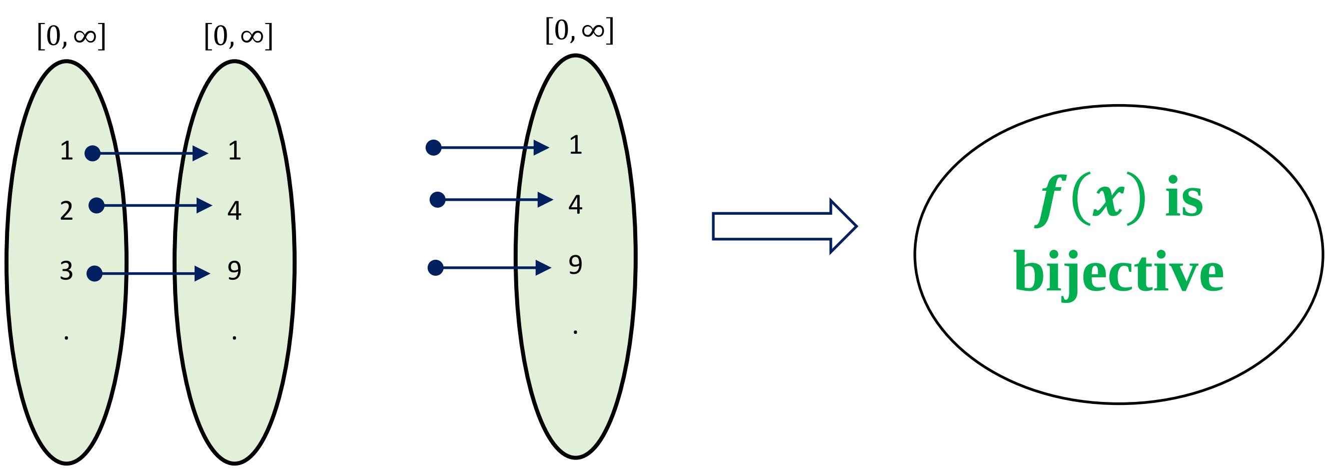
Theorem: (Invers function theorem)
Suppose that $f: A \to B$ is bijective function, where A and B are intervals. If f is diff. at $a \in A$ and $f'(a) \neq 0$ then $f^{-1}$ is diff at $b = f(a)$ and $(f^{-1})' = \frac{1}{f'(a)}$

**Remember:** not any function has Invers. To be has Invers it must be bijective (تقابل). That is mean the function is (1-1 and onto)
**1-1** is every element in Doman joint in one element in codomain
**Onto** is every element in codomain joint

**Example**: Let $f: [0, \infty) \to [0, \infty) s.t\ f(x) = x^2$ ? f is 1 – 1 and onto, then $f^{-1}$ exists.
Solution:

$\because y = x^2, \to x = \sqrt{y}$ , $f: [0, \infty) \to [0, \infty)$ and $f^{-1}(y) = \sqrt{y}$.

$\forall y, \left(f^{-1}(y)\right)' = \frac{1}{f'(x)} = \frac{1}{2x} = \frac{1}{2\sqrt{y}}$.

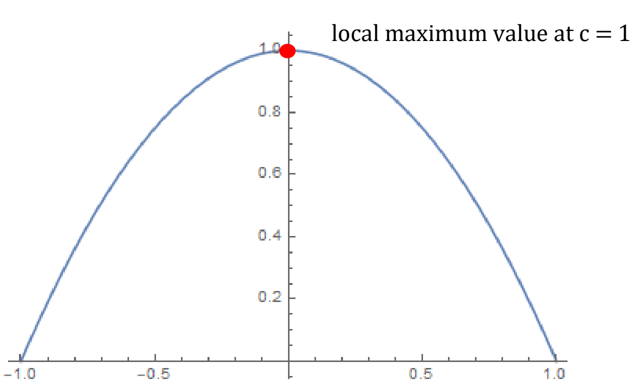


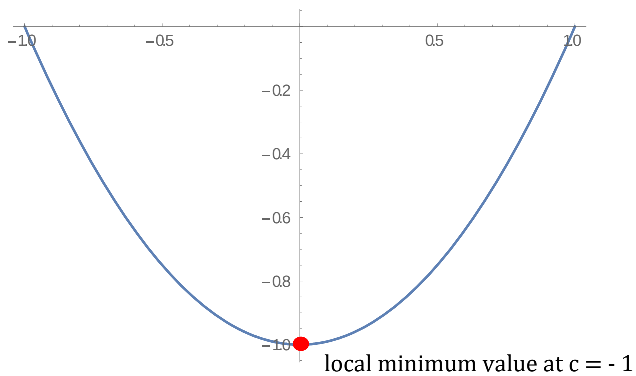

Condition of 1-1

Condition of onto

**Remember:** f has Invers **iff** f is bijective

Definition: A function f has a local maximum value at c, if $\exists\, J$ (J is open interval), $C \in J$ , s.t $f(x) \leq f(c), \forall x \in J$.
A function f has a local minimum value, if $\exists\, J$ ( J is open interval ), $C \in J$ s.t $f(c) \leq f(x), \forall x \in J$ .



Theorem: (Local Extreme Value)
If f is diff. at c and has local Max. (Or min.) value at c, then $f'(c) = 0$
Proof:

Suppose f has min. value at c → by def. ∃ $J$ ( J is open interval ),
$C \in J, s.t\ f(c) \leq f(x), \forall x \in J$.
Then $f(x) - f(c) \geq 0$ . there are two cases:

1. If $x > c \to x - c > 0 \ \Rightarrow \frac{f(x)-f(c)}{x-c} \geq 0 \to f_t'(c) = \lim_{x-c^+} \frac{f(x)-f(c)}{x-c} > 0$
2. If $x < c \to x - c < 0 \ \Rightarrow \frac{f(x)-f(c)}{x-c} \leq 0 \to f_t'(c) = \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} \leq 0$

   Since f is did at c, then $0 \leq f_+'(c) = f'(c) = f_-'(c) \leq 0 \to f'(c) = 0$

**Local maximum or minimum**

1. The function must be diff.
2. Either has local maximum Or has local minimum
3. First Derivative when equal zero solve this equation to find the local point
4. Put the local point in Derivative to know type of local it must be either greater than 0 or less than zero
5. When you put local point and have got 0 find the next derivative to get negative or positive number
6. Positive $f(c) > 0$ is local Minimum point or negative $f(c) < 0$ Local maximum

Lecher 6:

Remark: The converse of the above theorem is not true
**Example**: Let $f: R \to R$, s.t $f(x) = x^3$ diff on R$f'(x) = 3x^2 \to x = 0$

Theorem: (Roll's theorem)
If f is diff on $(a, b)$ and Continuous $[a, b]$ and $f(a) = f(b)$, then there exists c, $a < c < b$, such that $f'(c) = 0$
Proof:

Since $[a, b]$ is compact and f cont. on $[a, b]$, then f has max. and min → $\exists c_1, c_2 \in [a, b]$ s.t $f(c_1) =$ max. Value $f(c_2)$ min. Value.
Now, if $f(c_1) \neq f(c_2) \ \Longrightarrow$ at last one of $c_1$ and $c_2$ is not equal to a or b → f has a local max. or min (or both) in $[a, b]$.
By theorem ( … ), $f'(c_1)$ or $f'(c_2)$ (or both) equal to zero

**Example**: find the value of c, witch satisfy Roll's theorem, where $f:[-1,1] \to R$, s.t $f(x)=\sqrt{1-x^2}$.
Solution:

$1-x^2 \geq 0 \to -x^2 \geq -1 \to$
$x^2 \leq 1 \to -1 \leq x \leq 1$
$\therefore$f is diff on $(-1,1)$ and continuous on $[-1,1]$
$a=-1, b=1$, then $f(-1)=0=f(1)$
by Roll's theorem,
$\exists c, -1<c<1 \; s.t \; f'(c)=0$
$\because f'(x)=-\frac{2x}{2\sqrt{1-x^2}}$ and $f'(c)=0 \to c=0$

**Find c satisfy Roll's theorem**

1. Find Rang of function by differential rules
2. Make sure that the function is differentiable and continuous over the given interval
3. $f(a)=f(b)$
4. by Roll's theorem
   $\exists c \in a \leq c \leq b$
   s.t $f'(c)=0$
5. find $f'(x)$
6. change all x by c in $f'(x) \to f'(c)$
7. now $f'(c)=0$ to find c

Theorem: (Mean Value theorem) (M.V.T)
If f is diff. on $(a,b)$ and continuous on $[a,b]$, then there exists c, $a<c<b$, such that
$f'(c)=\frac{f(b)-f(a)}{b-a}$

Proof:

Define $g(x)=f(x)-\lambda x$ , $\lambda=\frac{f(b)-f(a)}{b-a}$
Since f continuous on $[a,b]$ and $\lambda x$ continuous on $[a,b]$, them g is continuous on $[a,b]$
Also, f diff on $(a,b)$ and $\lambda x$ diff. on $(a,b)$, there g is diff. on $(a,b)$.
$g(a)=g(b) \to \exists c \in (a,b)$ s.t $g'(c)=0$
Since $g'(x)=f'(x)-\lambda \to g'(c)=f'(c)-\lambda=0$
$\to f'(c)=\lambda$, then $f'(c)=\frac{f(b)-f(a)}{b-a}$

**Example:** Let $f:[0,1] \to R, f(x)=\sqrt{x}$ , find the Value of c which satisfy M.V.T.
Solution:

f is continuous on $[0,1]$ and diff. on $(0,1)$ and $f'(x)=\frac{1}{2\sqrt{x}}$.
$\frac{f(b)-f(a)}{b-a}=\frac{1-0}{1-0}=1 \Rightarrow f'(c)=\frac{1}{2\sqrt{c}}=1 \to \frac{1}{\sqrt{c}}=2$
$c=\frac{1}{4}$

Corollary: if f is continuous on $[a,b]$ and diff. on $(a,b)$, $f'(x)=0, \forall x \in (a,b)$, then f is constant.
Proof:

Let $x_1, x_2 \in [a,b]$ and $x_1<x_2$, since f is continuous
Since f is continuous on $[a,b]$, then f continuous on $[x_1,x_2]$ and f is diff. on $(a,b)$,
So, f is diff on $(x_1,x_2)$.
by M.V.T, $\exists c$ , $c \in (x_1,x_2)$ s.t $f'(c)=\frac{f(x_2)-f(x_1)}{x_2-x_1}=0$
so $f(x_2)-f(x_1)=0 \Rightarrow f(x_2)=f(x_1)$ for any $x_1, x_2$ in $[a,b]$, then f is constant function.

Corollary: if f continuous on $[a, b]$, diff. on $(a, b)$ and $f'(x) \geq 0$ , $\forall x \in (a, b)$, then f is an increasing [f is an increasing if $x_1 < x_2$,then $f(x_1) \leq f(x_2)$]

Corollary: if f continuous on $[a, b]$, diff. on $(a, b)$ and $f'(x) \leq 0$ , $\forall x \in (a, b)$, then f is an decreasing [f is an decreasing if $x_1 < x_2$,then $f(x_2) \leq f(x_1)$]

Definition: if f is n-time diff. and $f^{(n)}$ is continuous function then f is called n-times continuously differentiable.

Taylor's theorem: suppose $\sum_{n=0}^{\infty} a_n x^n$ is a power series with radius of Convergence R, then $|x| < R$.

i.e.: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ then $f'(x) = \sum n\, a_n x^{n-1}$ , $|x| < R$.

Lecher 7:

Definition: let f be n-times continuous diff. Function at 0, the Taylor polynomial of degree n for f at 0 $\left(f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + .. + \frac{f^n(0)}{n!}x^n\right)$.

Is definition by $T_n \left(f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + .. + \frac{f^n(0)}{n!}x^n\right)$

**Example:** let $f(x) = e^x$ , $f^{(r)}(x) = e^x$ , $r = 1,2, \ldots$

$f^{(n)}(0) = 1$ , $\forall r$.

$T_0\, f(x) = 1$

$T_1\, f(x) = f(0) + \frac{f'(0)}{1!}x = 1 + x$

$T_2\, f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{x^2}{2}$

نلاحظ ان متعددة حدود تايلور تقترب من الدالة $e^x$ بالقرب من النقطة $x = 0$

ساكنة

Definition: the point x is called stationary if $f'(0) = 0$

Theorem: Classification theorem for local extrema

If f is (n+1)-times continuously diff on open interval contains a and $f^{(r)}(a) = 0$ , $r = 1,2, \ldots, n$ and $f^{(n+1)}(a) \neq 0$, then

1. n+1 even, $f^{(n+1)}(a) > 0$ , $\rightarrow x = a$, local min. of f.
2. n+1 even, $f^{(n+1)}(a) < 0$ , $\rightarrow x = a$, local max. of f.
3. n+1 odd, f has neither a local max. n or local min. at $x = a$

**Example:** determine the nature of the stationary point of $f(x)$ where $f(x) = x^6 - 6x^4$

Solution:

Since f is polynomial function, then f infinitely diff.

$f'(x) = 6x^5 - 24x^3$

$= 6x^3(x^2 - 4) = 6x^3(x-2)(x+2) = 0$

$\therefore f$ has stationary point at $x = 0$ , $x = 2$ , $x = -2$

$f''(x) = 30x^4 - 72x^2$

$f''(\pm 2) = 192 > 0$

So f has a local min at $x = \pm 2$

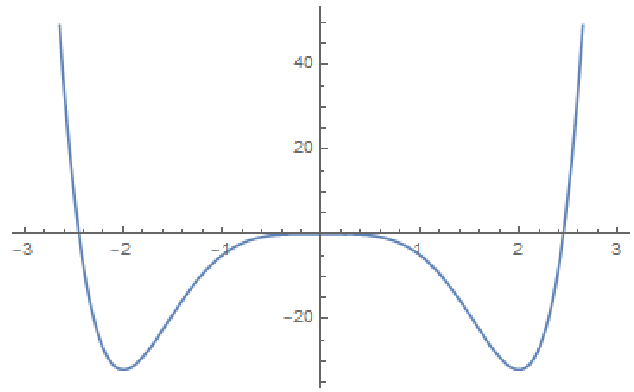
$f^{(3)}(x) = 120x^3 - 144x$

$f^{(3)}(0) = 0$

So $f^{(4)} = 360x^2 - 144$

$f^{(4)}(0) = -144 < 0$

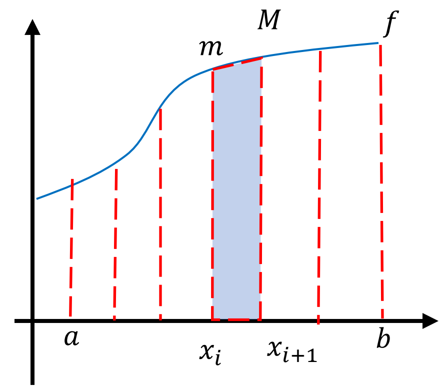
The f has a local max at $x = 0$



# Chapter Seven

# Reiman Integral

## تكامل ريمان

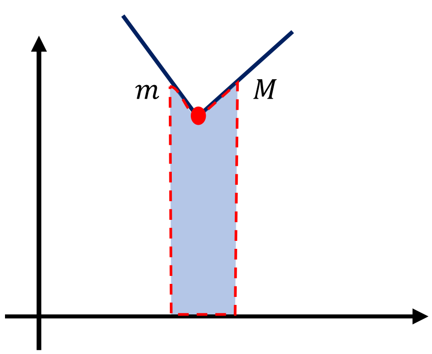



ان فكرة ريمان هو حساب المساحة للمنطقة

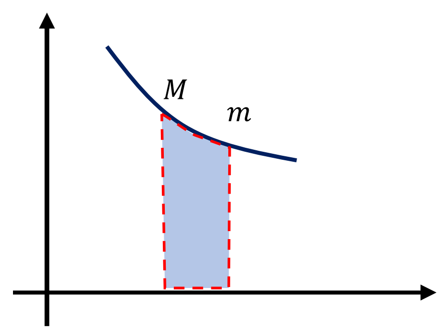
$$(1)\ \dots\dots\dots\dots \begin{cases} x_{i+1} - x_i = \text{فأذا كان العرض} \\ m_i = f(x_i) = \text{الطول} \end{cases}$$

$$(2)\ \dots\dots\dots\dots \begin{cases} x_{i+1} - x_i = \text{او اذا كان العرض} \\ M_i = f(x_{i+1}) = \text{الطول} \end{cases}$$

فان مجموع المساحات في (1) يسمى بمجموع ريمان الأدنى. ومجموع المساحات في (2) يسمى مجموع ريمان الأعلى اما اذا كانت المستطيلات في التجزئة انعم فان المجموع (1) يقترب من المجموع (2). مع ملاحظة انه قد لا تكون اعلى قيمة او اقل قيمة للدالة هي صورة احدى رؤوس الفترة $[x_i , x_{i+1}]$ كما في الشكل التالي:




Or




## Reiman Integrable:

Let f be a bounded function defined on $[a , b] = J$, where $|J| = b - a$ is the **length** of J. if $\Pi = \{a = x_0 , x_1 , x_2 , \dots , x_n = b\}$ is **ordered** subset of J where the first element is **a** and the last is **b**, then $\Pi$ is called the **partion** of J

And $J_i = [x_i , x_{i+1}] , i = 0 , 1 , 2 \dots , n - 1$ are called the subintervals of J.

$\Pi: x_0 < x_1 < x_2 < \cdots < x_n$

Since f is bounded function, So

$M = \sup\{f(x): x \in J\}$ , $M_i = \sup\{f(x): x \in J_i\}$

$m = \inf\{f(x): x \in J\}$ , $m_i = \inf\{f(x): x \in J_i\}$

**To Make Partion for Interval**

1. Make set as $\Pi = \{ a , \dots , b\}$
2. $a$: is first (Start Interval)
3. $b$: is second (End Interval)
4. Numbers between a and b are Order

- Lower sum of f relative to Π.

$$\underline{R}(f,\Pi) = \sum_{i=0}^{n-1} m_i|J_i|$$
$$= \sum_{i=0}^{n-1} m_i(x_{i+1} - x_i)$$

**Don't forget**
$J$ : is Main interval
$J_i$ : is Subinterval

- Upper sum of f relative to Π

$$\overline{R}(f,\Pi) = \sum_{i=0}^{n-1} M_i|J_i|$$
$$= \sum_{i=0}^{n-1} M_i(x_{i+1} - x_i)$$

Lecher 8:

**Remark:** $\forall \boldsymbol{i}, \boldsymbol{m} \le \boldsymbol{m_i} \le \boldsymbol{M_i} \le \boldsymbol{M}$

Proof:

If $A = \{f(x) : x \in J\}$
$B = \{f(x) : x \in J_i\} \implies B \subseteq A$
By So, $\inf(A) \le \inf(B)$ (properties of inf , sup).
$\implies m \le m_i, \forall i$
And $\sup(B) \le \sup(A)$, then $M_i \le M, \forall i$
Also, Since $\inf(B) \le \sup(B) \Rightarrow m \le m_i \le M_i \le M$.

**Remember**
**f** is bounded iff $f(x)$ is bounded
(**Has Upper & Lower bound**)

Remark: if f is bounded function defined on $[a,b]$, then

$$m(b-a) \le \underline{R}(f,\Pi) \le \overline{R}(f,\Pi) \le M(b-a)$$

Proof:

Since $m \le m_i \le M_i \le M$ (by above Remark).
$\Rightarrow m\,|J_i| \le m_i|J_i| \le M_i|J_i| \le M|J_i|, \forall i$
$\implies \sum_{i=0}^{n-1} m\,|J_i| \le \sum_{i=0}^{n-1} m_i\,|J_i| \le \sum_{i=0}^{n-1} M_i\,|J_i| \le \sum_{i=0}^{n-1} M\,|J_i|$
$\Rightarrow m\sum_{i=0}^{n-1} |J_i| \le \underline{R}(f,\Pi) \le \overline{R}(f,\Pi) \le M\sum_{i=0}^{n-1}|J_i|$
$\Rightarrow m(b-a) \le \underline{R}(f,\Pi) \le \overline{R}(f,\Pi) \le M(b-a)$

**Definition:** Let $\Pi, \Pi^*$ be two partion of $[a,b]$, if $\Pi \subseteq \Pi^*$ then $\Pi^*$ is called the refinement of $\Pi$.

Example: Let $\Pi = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ be a partion of $[0,1]$ then $\Pi^* = \left\{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\right\}$ also partion of $[0,1]$ and and $\Pi \subseteq \Pi^*$

Remark: if f is bounded function on $[a,b]$ and $\Pi^*$ is refinement of $\Pi$, then.

1. $\underline{R}(f,\Pi) \le \underline{R}(f,\Pi^*)$
2. $\overline{R}(f,\Pi^*) \le \overline{R}(f,\Pi)$
3. $\underline{R}(f,\Pi) \le \overline{R}(f,\Pi^*)$

Now, $\overline{R}\,(f) = \{\,\overline{R}\,(f\,,\Pi) :\ \Pi\ is\ partion\} \neq\ \phi$
And, $\underline{R}\,(f) = \{\,\underline{R}\,(f\,,\Pi) :\ \Pi\ is\ partion\} \neq\ \phi$
Since ∃ a partion Which is $[a,b]$ itself.

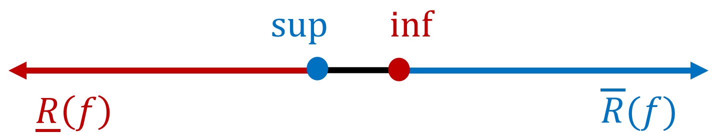
So, there exist at least one $\underline{R}(f,\Pi)$ and at least one $\overline{R}\,(f,\Pi)$

**Also**, since $\underline{R}\,(f,\Pi) \leq\ \overline{R}(f,\Pi^*)$, there any element in $\overline{R}$ will be upper bounded of $\underline{R}$. And any element in $\underline{R}$ will be lower bounded of $\overline{R}(f)$. So

$\overline{\int}\ f =\ R\overline{\int}\ f = \inf(\overline{R}\,,f)$ تكامل ريمان الأعلى لـــ f

$\underline{\int}\ f = R\underline{\int}\ f = \sup(\underline{R}\,,f)$ تكامل ريمان الأدنى لـــ f

$\underline{\int}\ f \leq \overline{\int}\ f$




Definition: Let f be a bounded function definition on $[a,b]$ f is Reiman integrable iff $\underline{\int}\ f = \overline{\int}\ f$, and denoted by $R\int\ f\ \ or\ \int_a^b f$.

**Example**: find $\int_0^2 f(x)\ , s.t\ f(x) = 3x$.

Solution:

$$[a,b] = [0,2] = J$$
$$|J| = b - a = 2 - 0 = 2.$$
$$\Pi = \left\{a < a + \frac{b-a}{n} < a + 2\frac{b-a}{n} < \cdots < a + n\frac{b-a}{n} = b\right\}$$
$$\left\{0 < 0 + \frac{2+0}{n} < 0 + 2\,.\frac{2}{n} < \cdots < 0 + n\frac{2}{n} = b\ \right\}$$
$$0 < \frac{2}{n} < 2\,.\frac{2}{n} < \cdots < 2$$

$$0 < \frac{2}{n} < \frac{4}{n} < \cdots < 2$$
$$\Downarrow\quad \Downarrow\quad \Downarrow\qquad\quad \Downarrow$$
$$x_0\quad x_1\quad x_2\qquad\quad x_n$$

i.e.: $0 < \frac{2}{n} < \frac{4}{n} < \frac{6}{n} < \frac{8}{n} < \cdots < 2$

The sub intervals are $J_0 = \left[0\,,\frac{2}{n}\right]$

$$J_1 = \left[\frac{2}{n}, \frac{4}{n}\right]$$
$$J_2 = \left[\frac{4}{n}\ , \frac{6}{n}\right]$$
$$\vdots$$
$$j_{n-1} = \left[\frac{2(n-1)}{n}\ , 2\right]$$
$$|J_i| = \frac{2}{n}\ \ , \forall i.$$

**Remember**

$|J_i| = b_i - a_i$

as: $2 - \frac{2(n-1)}{n} = \frac{2n-2n+2}{n} = \frac{2}{n}$

الان نجد المجموع من الأدنى

$$\underline{R}\,(f,\Pi) = \textstyle\sum_{i=0}^{n-1} m_i |J_i|$$

$= m_0(x_1 - x_0) + m_1(x_2 - x_1) + \cdots + m_{n-1}(x_n - x_{n-1})$
$= f(x_0)\frac{2}{n} + f(x_1)\frac{2}{n} + \cdots + f(x_{n-1}) . \frac{2}{n}$
بما ان الدالة متزايدة فان صورة بداية الفترة دائما هي اقل الصور
$= 0 . \frac{2}{n} + \frac{6}{n} . \frac{2}{n} + \frac{12}{n} . \frac{2}{n} + \cdots + \frac{6(n-1)}{n} . \frac{2}{n}$
$= 0 + \frac{12}{n^2} + \frac{24}{n^2} + \cdots + \frac{12}{n^2}(n-1)$
$= \frac{12}{n^2}(1 + 2 + \cdots + n - 1)\ \left(\sum_{x=1}^{n} x = \frac{n(n-1)}{2}\right)$
$= \frac{12}{n^2}\left(\frac{n(n-1)}{2}\right)$
$= \frac{6(n-1)}{n} = \frac{6n}{n} - \frac{6}{n} = 6 - \frac{6}{n}$

$\underline{\int} f = \sup[\underline{R}(f, \Pi)] = \sup\left[6 - \frac{6}{n} : n \in N\right] = 6$ تكامل ريمان الأسفل

الان نجد المجموع الأعلى ثم نكامل من الأعلى

Now,
$\overline{R}(f, \Pi) = \sum_{i=0}^{n-1} M_i |J_i|$
$= M_0\ (x_1 - x_0) + M_1(x_2 - x_1) + \cdots + M_{n-1}(x_n - x_{n-1}).$
$= f(x_1). \frac{2}{n} + f(x_2) . \frac{2}{n} + \cdots + f(x_n) . \frac{2}{n}$
$= \frac{6}{n} . \frac{2}{n} + \frac{12}{n} . \frac{2}{n} + \cdots + \frac{6n}{n} . \frac{2}{n}$
$= \frac{12}{n^2} + \frac{24}{n^2} + \cdots + \frac{12n}{n^2}$
$= \frac{12}{n^2}(1 + 2 + \cdots + n) . \left(\sum_{x=1}^{n} x = \frac{n(n+1)}{2}\right)$
$= \frac{12}{n^2} . \frac{n(n+1)}{2} = \frac{6(n+1)}{n} = 6 + \frac{6}{n}\ , \forall n$
$\overline{\int} f = \inf[\overline{R}\ (f, \Pi)] = \inf\left[6 + \frac{6}{n} : n \in N\right] = 6$
$\therefore \overline{\int} f = \underline{\int} f = 6 = \int_0^2 f(x) = 6$

بما ان تكامل ريمان من الأعلى يساوي تكامل ريمان من الأسفل فالدالة قابلة للتكامل الريماني

Not:

$$\sum_{x=0}^{n} x^2 = \frac{n(n+1)(2n+1)}{6}$$

Lecher 9:

Theorem: if f is bounded function defined on $[a, b]$ then f is R.I iff $\forall \epsilon > 0$ , $\overline{R}(f, \Pi)\ \underline{R}(f, \Pi) < \epsilon$.

Theorem: Let f is a continuous function on $[a, b]$, then f is Reiman integrable.

Proof:

Let $\epsilon > 0$ , since f is continuous on $[a, b]$, then f is uniformly continuous, then $\exists\ \delta > \epsilon$ s.t $\forall\ x, y \in [a, b]$.
$|f(x) - f(y)| < \frac{\epsilon}{b-a}$
By Arch. Property $\exists n \in N$ s.t $n\delta > b - a \implies \frac{b-a}{n} < \delta$.
$\implies \Pi = \{a = x_0\ , x_1\ , \ldots . , x_n = b\}$ be a partion of $[a, b]$.

s.t $|J_i| = \frac{b-a}{n}$, $J_i = [x_i , x_{i+1}]$
So, $|J_i| = |x_{i+1} - x_i| < \delta \ , \forall i$
f is bounded and has max , min. Values in $[a, b]$ . (Since f is continuous on $[a, b]$)
and in $J_i$ , $\forall i$
So, $\exists t_i , t_i^* \ in\ J_i$ s.t $M_i = \sup[f(x) : x \in J_i ] = f(t_i)$
$m_i = \inf[f(x) \ : x \in J_i] = f(t_i^*)$
Since $|t_i - t_i^*| < \delta$ , then $|f(t_i) - f(t_i^*)| < \frac{\epsilon}{b-a}$

$$\therefore \ \overline{R}\,(f, \Pi) - \underline{R}(f, \Pi) = \sum_{i=0}^{n-1} M_i|J_i| - \sum_{i=0}^{n-1} m_i|J_i|$$
$$= \sum\big(f(t_i) - f(t_i^*)\big)\ |J_i|$$
$$= \ \sum_{i=0}^{n-1} \frac{\epsilon}{b-a}\ \ |J_i|$$
$$= \frac{\epsilon}{b-a}\ \sum_{i=0}^{n-1}|J_i| = \frac{\epsilon}{b-a}\ .(b-a) = \epsilon$$

So f is R.I

* **The Convers of above theorem is not true.**

**Example:** Let $f(x) = \begin{cases} 3 & if\ x \le 0 \\ 7 & if\ x > 0 \end{cases}$ ,defined on $[-5,5]$, f is Reiman integrable but not continuous.

Solution:

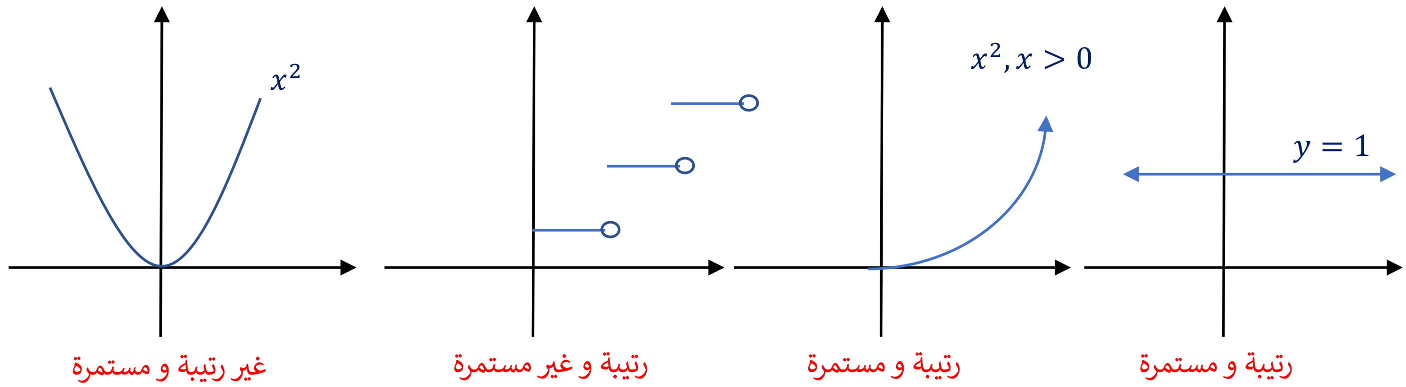
$$\Pi = \left\{-5\,, -\frac{1}{n}\,, \frac{1}{n}\,, 5\right\} , \forall n.$$
$$J_0 = \left[-5\,, -\frac{1}{n}\right]$$
$$J_1 = \left[-\frac{1}{n}, \frac{1}{n}\right]$$
$$J_2 = \left[\frac{1}{n}\,, 5\right]$$
$$\underline{R}(f, \Pi) = \sum_{i=j}^{n-1} m_i|J_i| = m_0|J_0| + m_1|J_1| + m_2|J_2|$$
$$= f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + f(x_2)(x_3 - x_2)$$
$$= 3\left(-\frac{1}{n} - (-5)\right) + 3\left(\frac{1}{n} - \left(-\frac{1}{n}\right)\right) + 7\left(5 - \frac{1}{n}\right)$$
$$= -\frac{3}{n} + 15 + \frac{3}{n} + \frac{3}{n} + 35 - \frac{7}{n}$$
$$= 50 - \frac{4}{n}$$
$$\underline{\int} f = \sup\left(\underline{R}(f, \Pi)\right)$$
$$= \sup\left[50 - \frac{4}{n} : n \in N\right] = 50$$
$$\overline{R}\,(f, \Pi) = \sum_{i=0}^{n-1} M_i|J_i|$$
$$= M_1|J_0| + M_1|J_2| + M_3|J_2|$$
$$= M_0(x_1 - x_0) + M_1(x_2 - x_1) + M_2(x_3 - x_2)$$
$$= f(x_1)(x_1 - x_0) + f(x_2)(x_2 - x_1) + f(x_3)\ (x_3 - x_2)$$
$$= f(x_1)(x_1 - x_0) + f(x_2)(x_2 - x_1) + f(x_3)(x_3 - x_2)$$
$$= 3(-\frac{1}{n} - (-5) + 7\left(\frac{1}{n} - \left(-\frac{1}{n}\right)\right) + 7\left(5 - \frac{1}{n}\right)$$
$$= \frac{3}{n} + 15 + \frac{7}{n} + \frac{7}{n} + 35 - \frac{7}{n}$$
$$= -\frac{3}{n} + 15 + \frac{7}{n} + 35$$

$= 50 + \frac{4}{n}$

$\overline{\int} f = \inf\left(\overline{R}(f,\Pi)\right) = \inf\left[50 + \frac{4}{n}\right] = 50$

$\therefore \overline{\int} f = \underline{\int} f \Rightarrow \int_{-5}^{5} f(x) = 50$

**Theorem: if f is bounded monotonically (non-decrasing) on $[a,b]$, Then f is R.I**




Proof:

Let $\epsilon > 0$ , $\Pi = \{a = x_0 , x_1 , \dots x_n = b\}$ be a partion on $[a,b]$,such that

$J_i = [x_i , x_{i+1}]$ and $|J_i| = \frac{b-a}{n}$

Since f is non-decreasing, then $M_i = f(x_i + 1)$ and $m_i = f(x_i)$

$\overline{R}(f,\Pi) = \sum_{i=0}^{n-1}[f(x_{i+1}) - f(x_i)]\,|J_i| \;\forall i$

$= [f(b) - f(a)]\left(\frac{b-a}{n}\right)$

By Arch. Property ,$\exists n \in N$ , such that $n\epsilon > \left(f(b) - f(a)\right)(b-a)$

$\Rightarrow \left(f(b) - f(a)\right)\frac{(b-a)}{n} < \epsilon$

Since $\overline{R}\;(f,\Pi) - \overline{R}\;(f,\Pi) = [f(b) - f(a)]\frac{b-a}{n} < \epsilon$

$\therefore\; \overline{R}\;(f,\Pi) - \underline{R}(f,\Pi) < \epsilon$

by theorem (**Let f be a continouos function on $[a,b]$, then f is R.I**)

Lecher 10:

**Theorem: if f is bounded and R.I on $[a,b]$, $f(x) \geq 0 \;\forall x \in [a,b]$. Then $\int f \geq 0$**

**Definition:** Let $S \subset R$, S is called negligible set, if there exisist a contable family open intervals $\{I_k\}$ such that

1. $S \subseteq \bigcup_{k=1}^{\infty} I_k$
2. $\forall\, \epsilon > 0 , \sum_{k=1}^{\infty}|I_k| < \epsilon$

نقول ان S مجموعة مهملة اذا امكن تغطيتها بطائفة $\{I_k\}$من الفترات المفتوحة والتي يقترب مجموع اطوالها من الصفر.

Example: Any finite set of reals is negligible.

Proof:

Let $S = \{x_1 , x_2 , \dots , x_n\} , \epsilon > 0 , \forall k , 1 \le k \le n$
Let $I_k$ be an open intervals with center $x_k$ and $|I_k| = \frac{\epsilon}{2n}$
So, $S \subset \bigcup_{k=1}^{n} I_k$
and $\sum_{k=1}^{n} |I_k| = \sum_{k=1}^{n} \frac{\epsilon}{2n} = \frac{\epsilon}{2n} \, n = \frac{\epsilon}{2} < \epsilon$
∴ S is negaligible set.

* **Any Countable of reals is negligible.**

Proof:

Let $\epsilon > 0 , \forall k , I_k$ open interval with center $x_k$
And $|I_k| = \frac{\epsilon}{2^{k+1}} = \epsilon \sum_{k=1}^{\infty} \frac{1}{2^{k+1}}$

هذه متسلسلة هندسية وان $r = \frac{1}{2}$ و $a = \frac{1}{2}$ لذا فان

$$\epsilon \sum \frac{1}{2^{k+1}} = \frac{\epsilon}{2} < \epsilon$$

* **Any subset of negligible set is negligible**
* **The union of family of negligible set is negligible**
* **Q is negligible (Since Q is countable)**
* **Any interval $[a, b]$ , $(a, b)$ , $[a, b)$ not negligible. (since its leuguth $= b - a$)**
* **R is not negligible, Since its length goes to ∞.**
* **Q' is not negligible**
* **There is an Uncountable negligible set which is Cantor set**

Theorem: (Lebesgu's Theorem for R.I)

مبرهنة ليبيك في التكامل الريماني.

Let f a bounded function on $[a, b]$, then f is Reiman Integrable if and only if the set dis continuous pionts of f is negligible set.

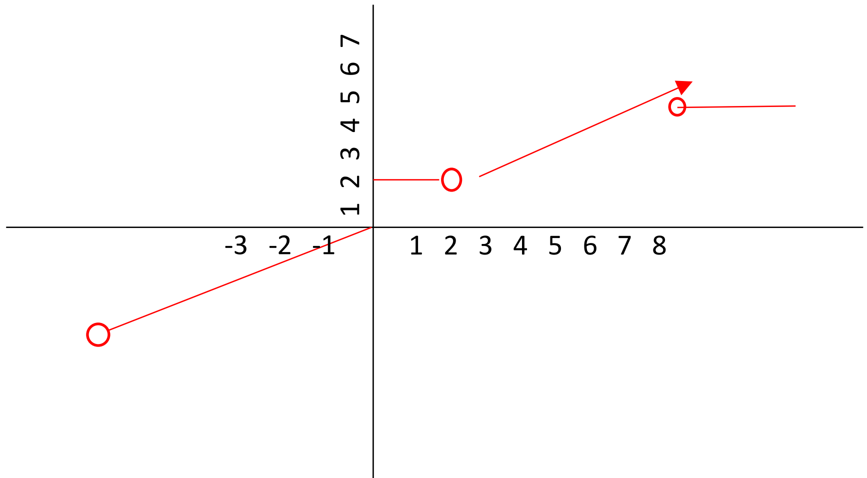
f قابلة للتكامل الريماني اذا وفقط اذا كانت مجموعة نقاط عدم الاستمرارية لـ f مجموعة مهملة.

**Example:** Use Lebesgu's theorem to show f is R.I, where $f: [-4,7] \to R$,

$$f(x) = \begin{cases} x & if -4 \le x < 0 \\ 2 & if\ 0 \le x < 2 \\ x + 1 & if\ 2 \le x \le 5 \\ 5 & if\ 5 < x \le 7 \end{cases}$$

Solution:

f is bounded since
$|f(x)| \le 6 , \forall x \in [-4,7]$
and the set of dis coutinuous point is $\{0 , 2 , 5\}$ is finite then it is negligble by "Theorem (Lebesgu's theoerm for R.I)". f is R.I




**Some Propertise of Rieman Integration:**

1. If f and g are Rieman Integrables on $[a,b]$ , $c \in R$ then $f+g$ , $c.f$ are R.I and
   I. $\int_a^b (f+g)dx = \int_a^b f(x)dx + \int_a^b f(x)dx$
   II. $\int_a^b (c\,.f)(x)dx = c\int_a^b f(x)dx$

   Proof:

   To prove that $f+g$ , $cf$ are $R.I$
   Since f is R.I, then the set of discontinuity of f is negliible set.
   and g is R.I, then the set of dis Continuity of g is negligible set
   (by Leb. Th. for R.I)
   So, the set of dis continuous point of $f+g$ is negligible set, then $f+g$ is R.I

Remark: Let R.I $[a,b]$ = the set of all Rieman integrable function on $[a,b]$, then by (i) and (ii), $\int_a^b$ is Linear transformation تحويل خطي

$$\int_a^b : R.I\ [a,b] \to R\ ,\int_a^b f \in R\ ,\forall\ f \in R.I\ [a,b]$$

Lecher 11:

2. If f is Reiman Integrable on $[a,b]$ , $f(x) \geq 0$ , $\forall x$ then $\int f \geq 0$

   Proof:

   Since $f(x) \geq 0$ , $\forall x \in [a,b]$
   Then $\underline{R}(f,\Pi) \geq 0$ $and$ $\overline{R}(f,\Pi) \geq 0$, for any partion $\Pi$ of $[a,b]$

$\Rightarrow \underline{\int} f \geq 0$ and $\overline{\int} f \geq 0$

Since f is R.I, then $\underline{\int f} = \overline{\int f} = \int f \geq 0$

3. If $f_1$ and $f_2$ are R.I on $[a,b]$ and, if $f_1 \geq f_2$, then $\int f_1 > \int f_2$
   Proof:

   Let $g = f_1 - f_2$

   $\therefore f_1(x) \geq f_2(x) \rightarrow (f_1(x) = f_1 \ , f_2(x) = f_2\ )$

   $f_1(x) - f_2(x) \geq 0\ , \forall x \in [a,b]$

   $\int g(x) \geq 0$ by proporty (2)

   $\int_a^b g(x) = \int_a^b f_1 - f_2 \geq 0$

   $= \int_a^b f_1 + (-f_2) \geq 0$

   $= \int_a^b f_1 + \int_a^b -f_2 \geq 0$ by (1) (i and ii)

   Then $\int f_1 \geq \int f_2$

4. If f is R.I , then $|f|$ is R.I and $|\int f| \leq \int |f|$
5. If f is R.I then $f^2$ is R.I
6. If f and g are R.I then the proudect f.g is R.I
7. $\int: R.I\ [a,b] \rightarrow R$ is not (one to one) function.
   We must prove that for some $f, g$ and $\int f = \int g$ but $f \neq g$ ?

   **Example:** let f , g $\in RI[-1,1]$ , $f: [-1,1] \rightarrow R$ , $g: [-1,1] \rightarrow$ we define f,g as function.

   $f(x) = x\ , \forall x$

   $$g(x) = \begin{cases} 0 & if\ x \neq 1 \\ 5 & if\ x = 1 \end{cases}$$

   Then $\int_{-1}^{1} f(x)dx = 0$ and

   $\int_{-1}^{1} g(x)dx = 0$

   but $f \neq g$

8. If f continuous on $[a,b]$ , $f(x) \geq 0\ , \forall x$ and $\int f(x) = 0$ then $f = 0$
   (mems $f(x) = 0\ , \forall x$)
   Proof:

Suppose $f(x) \neq 0$, then $\exists x \in [a,b]$ s.t $f(x) \neq 0$ , $y = f(x) > 0$ let $\left(\frac{y}{2}, \frac{3y}{2}\right)$ be open interval contains y.
Since f is continuos, then $\exists V$ open interval in R, such that $x \in V$ and
$f(V \cap [a,b]) \subseteq \left(\frac{y}{2}, \frac{3y}{2}\right)$
$\Longrightarrow V \cap [a,b] \subseteq f^{-1}\left(\frac{y}{2}, \frac{3y}{2}\right)$
$\Longrightarrow \exists I$ closed interval s.t $x \in I \subseteq [a,b]$ and $f(x) > 0 , \forall x$
And since f is continuous, then f has min.Value in I say
$m \Longrightarrow m = \min[f(x): x \in I]$
$\Longrightarrow m > 0$
$\Longrightarrow f(x) \geq 0$
$\Longrightarrow \int_a^b f \geq \int_I f \geq m|I| > a \to C!$ Since $\int f = 0$

Theorem: if f is R.I on $[a,b]$ and $F(x) = \int_a^x f(t)dt$, then F is Continuous on $[a,b]$

Proof:
Since f is R.I, then f is bounded on $[a,b]$ (by def).
$\Longrightarrow \exists M > 0 , s.t \ |f(x)| \leq M , \forall x \in [a,b]$
$|F(x) - F(c)| = \left|\int_a^x f(t)dt - \int_a^c f(t)dt\right|$
$= \left|\int_c^x f(t)dt\right| \leq \left|\int_c^x |f(t)|dt\right| \leq \left|\int_c^x M dt\right| = |M(x-c)| = M|x-c|$
Now, for given $\epsilon > 0$, Choose $\delta = \frac{\epsilon}{M}$, then if $|M(x-c)| < \delta$, then
$|F(x) - F(c)| \leq M|x-c| < M.\delta < \epsilon$
$\Longrightarrow F$ is continuous at c, $\forall x \in [a,b] \Rightarrow F$ is continuous on $[a,b]$.

Throrem: (Fumdamental Theorem of Calculus)

المبرهنة الأساسية في التفاضل.

If f is R.I on $[a,b]$ and $F(x) = \int_a^x f(t)dt$ and f is Continuous on $[a,b]$, then F is differentiable on (a,b) and $F' = f$

**Example:** Let f be R.I on $[a,b]$, if $F(x) = \int_a^x f(t)dt$ and F is diff on (a,b) s.t $F' = f$ show that $\int_a^b f(t)dt = F(b) - F(a)$

Theorem: (the integral Mean Value Theorem)

مبرهنة القيمة المتوسطة في التكامل.

Let f and g be continuous on $[a,b]$ with $g(x) \geq 0$ for $x \in [a,b]$ , then there is c $c \in [a,b]$

such that.

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

Proof:

Since f is Continuous on $[a,b]$, then f is bounded

$\Longrightarrow \exists m, M > 0\ , s.t\ m \le f(x) \le M\ , \forall x \in [a,b]$

Since $g(x) \ge 0 \Longrightarrow mg(x) \le f(x)g(x) \le Mg(x)\ , \forall x$

$$\Longrightarrow m\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le M\int_a^b g(x)dx$$

$\int_a^b g(x)dx = 0$

If $\int_a^b g(x)dx \Longrightarrow \int_a^b f(x)g(x) = 0 \Longrightarrow f(c)\int_a^b g(x)dx$

If $\int_a^b g(x)dx \ne 0 \Longrightarrow m\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le M\int_a^b g(x)dx$

$\Longrightarrow m \le \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \le M$

Apply (I.V.P) intemediate Value property on f and $[a,b]$ with.

$k = \frac{\int_a^b fg}{\int_a^b g} \Longrightarrow \exists c \in (a,b)\ s.t\ f(c) = k$

$$\therefore f(c) = \frac{\int_a^b fgdx}{\int_a^b gdx} \Longrightarrow \int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

Lecher 12:

# chapter eight

## Measure thery and Lebesgue's Integ ral

Let $I = (a, b)$ be open interval,We denotet. the length (Measure) of I by $\Delta(I)$ and $\Delta(I) =$ non-negative real number, then Δ is amapping on a family of open intervals in R.

Since every open set in R is the union of Cuntuble open intervals, (if G open in R ⇒ $G = UI_k$ ), then we Can define the length of an open set in R by the **sum** of the length of $I_k$ , $\forall k$.

ما معنى الطول (القياس) لاي مجموعة أخرى؟

وفي الحقيقة العالم الرياضي Lebsgue في عام 1941 – 1875 وجد مجموعة من المجموعة القابلة للقياس في هذه الحالة يمكننا القول ان توسيع مفهوم الاعداد الحقيقة ومفهوم الحجم للشكل المنتظم الى شكل غير منتظم.

### (1) The length of bounded opon interval.

let $I = (a, b)$ open interval and Δ= the set of all open boud Intervals in R. since $\mathcal{A} = \{I: I$ Ibounded open intervall}. we define the following meypping.

$$\Delta: \mathcal{A} \rightarrow R^+ , s.t\ \Delta(I) = \begin{cases} b - a & if\ I = (a, b) \\ 0 & if\ I = \emptyset \end{cases}$$

$\Delta(I)$ is cilled the length (measure) of I.

### " properties of " Δ

1. For any I, $\Delta(L) \geq 0$
2. If $I_1 \subseteq I_2$ , then $\Delta(I_1) \leq \Delta(I_2)$, $I_1, I_2 \in \mathcal{A}$
3. $\Delta(I_1 \cup I_2) \leq \Delta(I_1) + \Delta(I_2)$
4. If $I_1 \cap I_2 = \phi$ , thee $\Delta(I_1 \cup I_2) = \Delta C I_1 \mid + \Delta(I_2)$
5. If $I \subset \bigcup_{k=1}^{n} I_k$ , then $\Delta(I) \leqslant \sum_{k=1}^{n} \Delta(I_k)$
6. In general, if $I \subseteq \bigcup_{k=1}^{\infty} I_k$ , $then\ \Delta(I) \leqslant \sum_{k=1}^{\infty} \Delta(I_k)$
7. let $\delta \in R, I \in \mathcal{A}$. If we define the set $\delta + I = [\delta + x: x \in I]$, (translation of I, $|s|$units) then $\triangle (S + I) = \Delta(I)$

   **Proof :(1)**

   by definition, $0 \leq \Delta(I) = \begin{cases} \mathrm{b - a} \\ 0 \end{cases}$

   **proof:(2)**

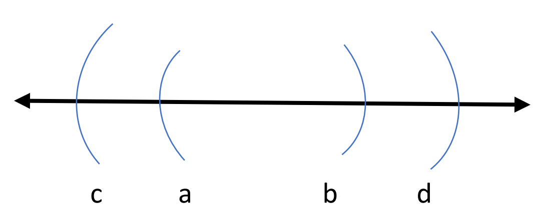
   let $I_1 = (a, b), I_2 = (c, d)$, and $I_1 \subseteq I_2$.

   $\Delta(I_1) \quad = (d - c) - (a - c) - (d - b)$

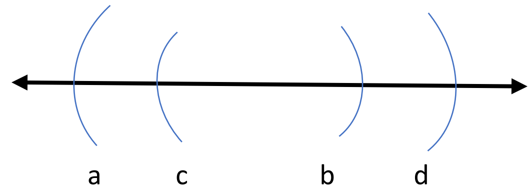
   $\qquad\quad = \Delta(I_2) - [(a - c) + (d - b)].$

   $\sin(a - c) + (d - c) \geqslant 0$, then

   $\Delta(I_1) \leqslant \Delta(I_2)$



**proof:(3)**




$I_1 = (a, b), I_2 = (c, d).$

$\Delta(I_1 \cup I_2) = b - a + (d - c) - (b - c)$

$= \Delta(I_1) + \Delta(I_2) - \Delta(I_1 \cap I_2) \geq 0$

$\leqslant \Delta(I_1) + \Delta(I_2).$

**proof:(4)**

If $I_1 \cap I_2 = \emptyset$, then $\Delta(I_1 \cup I_2) = \Delta(I_1) + \Delta(I_2) - \Delta(I_1 \cap I_2) = \Delta(I_1) + \Delta(I_2)$

**Proof:(5)** الاستقراء

The prove follows from induction and (2),(3)

**Proof:(6)**

let $I = (a, b)$, $I_k$ open interval, $k \leq 1,2, \ldots$

If $I \neq \phi \longrightarrow \Delta(I) = \sum_{k=1}^{\infty} \Delta([k)$

Since $I \subseteq \bigcup I_K$.

$\Rightarrow \{I_k : k = 1,2, \ldots\}$ open Cover for $I$.

let $\epsilon > 0$ and $I^* = (a - \epsilon, a + \epsilon)$ Cover for a.

$I^{**} = (b - \epsilon, b + \epsilon)$ Cover for b.

$\Rightarrow \bigcup_{k=1}^{\infty} I_k \cup I^* \cup I^{**}$ open Cover for $[a, b]$.

$\Rightarrow \bigcup_{k=1}^{\infty} I_k \cup I^* \cup I^{**}$ open Cover for $[a, b]$.

since [a,b] is Compact, then this Cover has a finite snbcover $\Rightarrow \exists I_1, I_{2,\ldots} I_n$ s.t

$I^* \cup I^{**} \bigcup_{k=1}^{n} I_k$ cover for $[a, b]$

$\Rightarrow I \subset \bigcup_{k=0}^{n} I_k \cup I^* \cup I^{**}$

$\Delta(I) \leq \Delta(I^*) + \Delta(I^{**}) + \sum_{k=1}^{n} \Delta(I_k)$

$= 2\epsilon + 2\epsilon + \sum_{k=1}^{n} \Delta(I_k)$

$= 4\epsilon + \sum_{k=1}^{n} \Delta(I_k)$

$\leqslant 4\epsilon + \sum_{k=1}^{\infty} \Delta(I_k)$

this is true for all $\epsilon > 0$, then $\Delta(I) \leqslant \sum_{k=1}^{\infty} \Delta(I_k)$

**Proof:(7)**

let $I = (a, b)$, then $s + I = (s + a, s + b)$

$\Delta(s + I) = s + b - (s + a) = b - a = \Delta(I)$

## (2) The length of Bounded open sets

اطوال المجموعات المفتوحة المقيدة

lemma: Let $G \subseteq R$, if $G$ is open set, then $G =$ union of Countable family of disjont bonnded open inlervals

اتحاد عناصر عائلة معدودة من الفترات المفتوحة المقيدة المنفصلة

Also, the representation of $G$ by this way will be unique
i.e: $\exists! \{I_i : I_i$ open bounded interval $, i = 1,2,\ldots\}$.
$s.t\ G = \bigcup_{i=1}^{\infty} I_i$ and $I_i \cap I_j = \emptyset, \forall_{i,j}$

**Definition**: Let $G$ be a bounded open set, then the length of $G, \Delta(G) = \sum_{i=1}^{\infty} \Delta I_i$, wheve $\{I_i\}$ is a unique family of open bounded intervals, s.t $G = \cup I_i$ and $I_i \cap I_j = \phi$.

**Quesion:** Does $\sum$ the Series $\sum_{i=1}^{\infty} \Delta I_i$ in the above definition is Convergent?
Yes, let $S_m = \sum_{n=1}^{m} \Delta I_n, \langle S_m \rangle$ the sequence of partial sums of $\sum \Delta I_n$, sine $G$ is bounded, then $\exists I$ open interval s.t $G \subset I$
$\Rightarrow \bigcup_{n=1}^{\infty} I_n \subset I \Rightarrow \sum_{n=1}^{\infty} \Delta I_n \leqslant \Delta(I) \Rightarrow S_m = \sum_{n=1}^{m} \Delta\,(I_n) \leqslant \sum_{n-1}^{\infty} \Delta I_n$
$\Rightarrow \langle S_m \rangle$ is bounded and sinc $\Delta(I_n) \geqslant 0$,
then $\langle S_m \rangle$ is non-decreusing $\Rightarrow \langle S_m \rangle$ Convergent.
i.e: $\sum_{n=1}^{\infty} \Delta I_n$ Converyent.

Lecher 13:

## Properties of Δ:

Let $\beta =$ the set of all bounded open sets
$= \{G : G\ bounded\ open\ set\}$

$$\Delta : \beta \to R^{+}\ s.t\ \Delta(G) = \begin{cases} \sum_{i=1}^{\infty} \Delta(I_i) & if\ G \neq \phi \\ 0 & if\ G \neq \phi \end{cases}$$

Then:

1. $\Delta(G) \geq 0$
2. If $G_1 \subset G_2$ ,then $\Delta(G_1) \leq \Delta(G_2)$
3. $\Delta(G_1 \cup G_2) = \Delta(G_1) + \Delta(G_2) - \Delta(G_1 \cap G_2)$
   and, if $G_1 \cap G_2 = \phi, then\ \Delta(G_1) + \Delta(G_2)$
4. If $\{G_n\}$ countable family, then $\Delta(\cup G_n) \leq \sum \Delta(G_n)$
   and if $\{G_n\}$ disjont, then $\Delta(\cup G_n) \leq \sum \Delta(G_n)$
5. $\Delta(S + G) = \Delta(G), S \in R$ ,where $S + G = \{s + x : x \in G\}$

**Remark:**

1. If G bounded , then $S + G$ is bounded
2. If G open, then $S + G$ is open
   Since the properties of a set does not chang by translation.
   Proof:
   1. G is bounded, then $\exists M > 0 \quad s.t\ d(x,y) = |x - y| \leq M$.
      $\Rightarrow d(s + x\ , s + y) = |s + x - (s + y)| = |x - y| \leq M \ \ , \forall x, y$
      $\therefore S + G$ bounded

   2. If G is open, then $G = \cup_{k=1}^{\infty} I_k \ \ , I_k$ is open
      interval and $I_k = (a_k\ , b_k)$
      $S + G = \cup(s + I_n)$
      $\quad\quad = \ \cup(S + a_k\ , s + b_k)$

## (3) Outer Measur of bounded sets

القياس الخارجي للمجوعة المقيدة

Defition: Let S bounded subset of R and $\theta(s) = \{G : G \ open\ bounded\ set\ and\ S \subset G\}$
than $M^*(s) = \inf\{\Delta(G): G \in \theta(s)\}$ is called the outer Measure of S.

القياس الخارجي للمجموعة S هو اصغر قياس لاصغر مجموعة تحوي S او اكبر قيد لمجوعة قياسات المجموعة المفتوحة التي تحوي S

**Remark:**

1. $M^*(s) =$ the Measure of the smallest open set G containg s.
2. $\forall\ (s) = \emptyset$, since s is bounded $\Longrightarrow\ \exists(a, b)$ open interval and $S \subseteq (a, b)$
3. The set $\{\Delta(G) : G \in \theta(s)\}$ is bounded below by Zero so inf **exists**

Example:

1. If S is bounded open set, then $M^*(s) = \Delta(s)$, since $M^*(s) = \inf\{\Delta(G) : G \in \theta(s)\}$
   since $s \in \theta(s)$ and $s \subseteq S$, then $\inf\{\Delta(G) : G \in \theta(s)\} = \Delta(s)$
2. $M^*(\emptyset) = \Delta(\emptyset) = 0$
3. If $S = \{x\}$, then $M^*(s) = 0$
   **Proof:**
   $\forall n \in N, G_n = \left(x - \frac{1}{n}\ , x + \frac{1}{n}\right)$ is open bounded interval and $S \subset \left(x - \frac{1}{n}\ , x + \frac{1}{n}\right) \forall n$
   $, \Delta(G_n) = \frac{2}{n} \to 0 \ \ as\ n \to \ \infty$
   then $M^*(s) = \inf\{\Delta(G_n)\ ; n \in N\} = \inf\left\{\frac{2}{n}\ ; n \in N\right\} = 0$
4. If S is countable set $M^*(s) = 0$.
   **Proof:**
   Let $S = \{x_1, x_2, x_3\ , \dots\}$ and $G_n = \left(X_n - \frac{\epsilon}{2^{n+1}}\ , X_n + \frac{\epsilon}{2^{n+1}}\right), \epsilon > 0 \to X_n \in G_n$
   $\to S \subset \cup\ G_n$
   $\Delta(G_n) = \frac{2\epsilon}{2^{n+1}} = \frac{\epsilon}{2^n}$
   $\Delta(G_n) \leq\ \sum_{n=1}^{\infty} \Delta(G_n) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon\ \left(\text{since } \sum_{n=1}^{\infty} \frac{1}{2^n} = 1\right)$ Geometric series.
   $M^*(s) = \inf\{\Delta(\cup\ G_n): n = 1,2, \dots\}$
   $\quad\quad = \inf\{\sum_{n=1}^{\infty} \Delta(G_n) : G \in \theta(s)\}$
   $\quad\quad = \inf\{\epsilon :\ \epsilon > 0\}\ = 0$

5. If S is negligible, then $M^*(s) = 0$
6. If S is bounded set and $M^*(s) = 0$, then S is negligible set

   Proof:

   Let $\epsilon > 0$ , $M^*(S) = 0 \to \inf\{\Delta(G): S \subset G\} = 0$

   $\to \ \forall \epsilon > 0 , \exists G \ \ s.t \ \ S \subset G \ and \ \Delta(G) < \epsilon$, $G$ is bounded open by Lemma $G = \bigcup I_n$ ,

   $I_i \cap I_l = \emptyset$ , $I$ open bounded interval $\to \ \Delta(G) = \sum_{n=1}^{\infty} \Delta(I_n) < \epsilon$

   $\therefore S \ \subseteq I_n$ , and $\sum \Delta(I_n) < \epsilon \to S$ is negligible
7. If $S = [a, b] \ or \ (a, b) \ or \ (a, b]$ , $then \ M^*(s) = b - a$

   **Proof:**

   Let $G_n = \left(a - \frac{1}{n} , b + \frac{1}{n}\right) \in N$

   $\Delta(G) = b - a - \frac{2}{n}$ , $n \in N$

   $S \subset \bigcup G_n$

   $\therefore M^* = \inf\{\Delta(G_n) : n \in\}$

   $= \inf\left\{b - a + \frac{1}{n} : n \in N\right\}$

   $= b - a$

## Properties of $M^*$

Let $C = \{S \subseteq R, S \ is \ bonuded\}$

$$M^*: C \to R^+ \ \ s.t \ \ M^*(s) = \begin{cases} \inf\{\Delta(G) : S \subset G & if \ S \neq \emptyset \\ 0 & if \ S = \emptyset \end{cases}$$

Than:

1. $M^*(s) \geq 0 \quad , S \subseteq C$
2. If $S_1 \subseteq S_2, then \ M^*(s_1) \leq M^*(s_2)$ ; $s_1, s_2 \in C$
3. $M^*(S_1 \cup S_2) \leq M^*(S_1) + M^*(S_2)$
4. If $\{S_n\}$ countable family s.t $S_n \in C$ , $\forall n$ , $then \ M^*(\bigcup S_n) \leq \sum M^*(S_n)$
5. If $t \in R$ , $S \in C$ then $M^*(t + s) = M^*(s)$

Lecher 14:

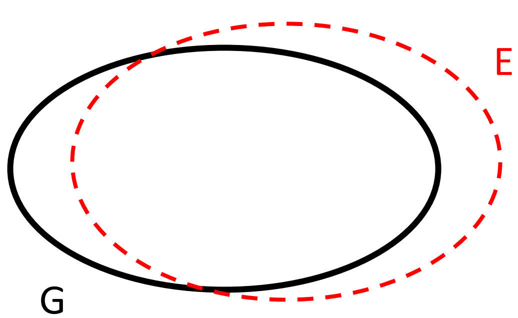
## (4) The measure of bounded setes:

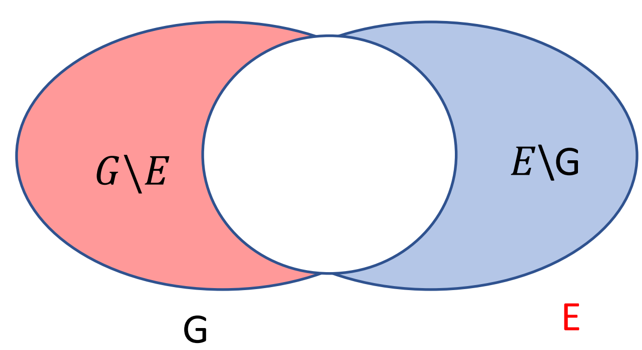
**Definition:** Let E is bounded subset of R, E is called measureable ( قابلة للقياس) set iff $\forall \epsilon > 0$ , $\exists G$,G is bounded open set, s.t $M^*|G - E| < \epsilon$

Where $|G - E| = \{x \in G \ and \ x \notin E\} \cup \{x: x \in E \ and \ x \notin G\}$

Or $|G - E| = G \backslash E \ \ \cup E \backslash G$

المجموعة E قابلة للقياس اذا امكن تقريبها لمجموعة مقيدة مفتوحة وعليه يكون الرسم اكثردقة كالاتي:



If E meusurable set then $M(E) = M^*(E)$ and $M(E)$ is called the maeasure of E.

Theorem: Let $E \subset E$ is bounded, E is called measurable iff $\forall \epsilon > 0$ , $\exists G$ bounded open set s.t $E \subset G$

Example:

1. If E bounded open then E is Meausurable and $M(E) = \Delta(E)$
   Proof:
   take$E = G \rightarrow \ |G - E| = \emptyset$
   $=M^* = |G - E| = 0 < \epsilon , \forall \epsilon$
   $\therefore E$ measurable.
2. If E is negligable, than measurable and $M(E) = 0$
   Proof:
   E is negligable than $\exists$ $\{I_n\}$ countable family of open interval set $E \subset \bigcup\{I_n\}$ and $\sum \Delta(I_n) < \epsilon$
   $\rightarrow \ \bigcup\{I_n\}$ open and bounded
   take $G_n = \bigcup I_n$
   $\therefore M^*(\bigcup I_n - E) < \epsilon \rightarrow E$ is measurable and $M^*(E) = M(E)$
3. If $E = \ [a,b] \ or \ (a,b) \ or \ [a,b] \ , then \ M(E) = b - a$
   Proof:
   If $E = [a,b]$
   Take $G = (a,b) \rightarrow G$ is open and bounded.
   $\therefore M^*(G - E) = M^*[a,b] = 0 < \epsilon , \forall \epsilon$
   $\therefore$E is Measurable and $M^*(E) = M(E) = b - a$

**Propertise of M:**

Let $D = \{E: E \ is \ bounded \ subset \ of \ R\}$

$M: D \rightarrow R^+$ s.t $M(E) = M^*(E)$ if E is measuerable

Then:

1. $M(E) \geq 0$
2. If $E_1 \subseteq E_2 \implies M(E_1) \leq M(E_2) \quad ; E_1, E_2 \in D$
3. $M(E_1 \cup E_2) = M(E_1) + M(E_2)$ if $E_1 \cap E_2 = \emptyset \rightarrow M(E_1 \cup E_2) = M(E_1) + M(E_2)$
4. $\{E_n\}$ countable family of bounded sets, the $M(\bigcup E_n) \leq \sum_n M(E_n)$

في تكامل ريمان ناخذ f دالة مقيدة كشرط أساسي لدالة التكامل. فان سنحاول نلخص هذا الشرط

**Definition:** let $S \subseteq R$ , $f: S \rightarrow R$ be a function, f is called Measurable, if $\forall G$ open set and $f^{-1}(G) \subseteq S, f^{-1}(G)$ must be measurable set.

**Remark**: when $f^{-1}(G) = S$, then f is mearable, if its domain is measurable set.

**Example**: Dose measurable function?

1. $f: [a, b] \to s.t\ f(x) = \begin{cases} 2 & , if\ x \in Q \\ 3 & , if\ x \in Q' \end{cases}$

   Let G open R, to find $f^{-1}(G)$?

   $$f^{-1}(G) = \begin{cases} \text{[a, b]} & \text{, if 2 and 3} \in \text{G} \\ \text{rationalin [a, b]} & \text{, if 2} \in \text{G} \\ \text{irrational in [a, b]} & \text{if 3} \in \text{G} \\ \emptyset & \text{if 2,3} \notin \text{G} \end{cases}$$

   $$\therefore M^*(f^{-1}(G)) = \begin{cases} \text{b} - \text{a} & \text{if 2,3} \in \text{G} \\ 0 & \text{if 2} \in \text{G (G is countable)} \\ \text{M}^*((\text{a, b})) = \text{b} - \text{a} & \text{if 3} \in \text{G} \\ 0 & \text{, if 2,3} \in \text{G} \end{cases}$$

   $\therefore f$ measurable

توضيح

$M^*((a-b)) - f^{-1}$

$= M^*(rational\ in\ [a, b] = 0$

$\therefore M^*(f^{-1}(G)) = M^* = b - a$

2. If $f: [a, b] \to R$ monotone then $f^{-1}(R) = [c, d]$ measurable $\Rightarrow f$ meausrable

## Lebesgue's Measure for unbounded sets

Defition: The set E is Lebesgue's Measurable, if for each A, we have

$$M^*(A) = M^*(A \cap E) \neq M^*(A \cap E^c)$$

**Remark:**

1. Since $A \subseteq (A \cap E) \cup (A \cap E^c) \Rightarrow M^*(A) \le M^*(A \cap E) + M^*(A \cap E^c)$
2. If E is Measurable, then $M^*(E)$ is denoted as $M(E)$

**Meaurable functions:** دوال القياس

**Definition:** Let f be a function defined on $[a, b]$, we call f be a maeasruble, if $\forall\ \alpha \in R$, then set $\{x: f(x) > \alpha\}$ is measurable

i.e.: f is measurable $\leftrightarrow f^{-1}(\alpha, \infty)$ mesurable set $\forall \alpha \in R$

Theorem: The function on $[a, b]$ is measurable iff one of the following conditions is hold

1. $\{x : f(x) \ge a\}$ is measurable, $\forall a \in R$
2. $\{x : f(x) > a\}$ is measurable, $\forall a \in R$
3. $\{x : f(x) \le a\}$ is measurable, $\forall a \in R$
4. $\{x : f(x) < a\}$ is measurable, $\forall a \in R$

Theorem: If f is measurable, then $|f|$ is measurable

Theorem: If f is measurable on $[a, b], r \in R$ then $(r + f), (r, f), (-f)$ are measurable

**Example:** Let $f(x) = \begin{cases} x + 5 & , if\ x < -1 \\ 2 & if\ -1 \leq x \leq 0 \\ x^2 & , if\ x > 0 \end{cases}$ show that f is measurable function.

Let $\alpha \in R$, then the set $\{x : f(x) \leq \alpha\}$

$$= \{x : f(x) \leq \alpha\} = \begin{cases} (-\infty, \alpha - 5) & , \text{if } \alpha < 0 \\ (-\infty, -5) \cup \{0\} & , \text{if } \alpha = 0 \\ (-\infty, \alpha - 5] \cup [0, \sqrt{\alpha}] & \text{if } 0 < \alpha < 2 \\ (-\infty, \alpha - 5] \cup [-1, \sqrt{\alpha}] & , \text{if } 2 \leq \alpha < 4 \\ (-\infty, \sqrt{\alpha}] & , \text{if } \alpha \geq 4 \end{cases}$$

**Example**: If $f(x) = c, \forall x \in R$, then f is meaurable since, if $\alpha \in R, c \geq \alpha$, then the set $\{x: f(x) \geq \alpha\} = R \rightarrow$ Measurable.

If $a \geq c$, then $\{x: f(x) \geq \alpha\} = \emptyset \rightarrow$ measurable.

Theorem: if $f_1$ and $f_2$ are mesurable, then $f_1 \pm f_2, f_1 . f_2, f_1/f_2\ f_2 \neq 0$ are measurable

**Example**: $g(x) = |x| + 2x$ is measurable

**Example:** Every continuous function is mearuable

Let $x \in R$, the set $\{x: f(x) > \alpha\} = f^{-1}(\alpha, \infty)$ by continuity $f^{-1}(\alpha, \infty)$ also open interval then f is measurable.

## Lebesgue Integral:

1. f bounded on $[a, b]$
2. f bounded on measurable set of function meaure
3. f unbounded
4. The general definition.

Thoerem: Every bounded Reman integral function on $[a, b]$ is Lebesgue integral and $R\int f = L\int f$

Proof:

If f is Rieman integral, then

$$R\int_{-a}^{a} f dx \leq R\int_{a}^{b} f\ dx = R\int_{a}^{b} f dx$$

by Remmark:

$$R\int_{a}^{b} f dx \leq L\int_{-a}^{b} f dx \leq L\int_{a}^{b} \leq R\int_{a}^{b} f\ dx$$

Then R.I $\Longrightarrow$ L.T but the Convers not true by the following example.

**Example:** Let $f: [0,1] \to R, f(x) = \begin{cases} 0 & if\ x \in Q \\ 1 & if\ x \in Q' \end{cases}$

P partion of $[0,1]$, $P = \{A_1, A_2\}$ where

$A_1 = \{x\ in\ [0,1]: x \in Q\} \Rightarrow M(A_1) = 0$

$A_2 = \{x\ in\ [0,1] : x \in Q'\} \Rightarrow M(A_2) = 1$ (since $A_2 = [0,1] \backslash A_1$)

$L(f, p) = \inf\{f(x)\} . M(A_1) + \inf\{f(x)\} . M(A_2)$

$x \in A_1$ $x \in A_2$

$= 0 . 0 + 1 . 1 = 1$

$U(f, p) = \sup\{f(x), M(A_1)\} + \sup\{f(x)\} . M(A_2)$

$x \in A_1$ $x \in A_2$

$= 0 . 0 + 1 . 1 = 1$

f is buonded function on $[a, b]$ and $P = \{A_1, A_2, \ldots, A_n\}$ be any partion of $[a, b]$ where $A_1, A_2, \ldots A_n$ are measurable subsets of $[a, b]$ such that

$$\bigcup_{n=1}^{n} A_i = [a, b]$$

And $M(A_i \cap A_j) = 0$ , $for\ i \neq j$

define $U(f, p) = \sum_{i=1}^{n} \sup_{x \in A_i} \{f(x)\} . M(A_i)$ L. upper sum

$L(f, p) = \sum_{i=1}^{n} \inf_{x \in A_i} \{f(x)\} . M(A_i)$ L. lower sum

Then $U(f, p) \geq L(f, p)$

$$L\int_{a}^{-b} f = \inf\ [\ U(f,p) : \ \forall p \text{ partion}\ ]$$

$$L\int_{-a}^{b} f = \sup\ [\ L(f,p) : \ \forall p \text{ partion}\ ]$$

f is called Lebesgue integrable, if

$$L\int_{-} f = L\int^{-} f = L\int_{a}^{b} f$$

**NOT**: $L\int_{-a}^{b} \le L\int_{a}^{-b}$

**Remark**: $R\int_{a-}^{b} \le L\int_{-a}^{b} \le L\int_{a}^{-b} \le \int_{a}^{-b}$

now, $\int_{-a}^{b} f = \sup\ [L(f,p)] = \inf\ [\ (U(f,p)] \ = \int_{a}^{-b} f$

$\therefore$ $f$ is Lebesgos integrable

Theorem: Let $f: [a,b] \to R$, bounded function, f is Leb. Integrable iff $\forall \epsilon > 0$ , $\exists$ a lebesgue portion P of $[a,b]$ such that $U(f,p) - L(f,p) < \epsilon$

Theorem: Every bounded measurable function f on $[a,b]$ is Lebesgue integrable on $[a,b]$